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Simple sandpile model of active-absorbing state transitions

Kavita Jain
Institut für Theoretische Physik, Universität zu Köln, 50937 Köln, Germany

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We study a simple sandpile model of active-absorbing state transitions in which a particle can hop out of a site only if the number of particles at that site is above a certain threshold. We show that the active phase has product measure whereas nontrivial correlations are found numerically in the absorbing phase. It is argued that the system relaxes to the latter phase slower than exponentially. The critical behavior of this model is found to be different from that of the other known universality classes.

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I. INTRODUCTION

Active-absorbing state transitions form an important class of nonequilibrium phase transitions whose best studied example is directed percolation (DP) [1]. A critical behavior different from DP is possible in the presence of additional symmetries as in parity-conserving DP [2]. Sandpile models in which the activity occurs when the local density exceeds a certain threshold can also exhibit such transitions, provided the total density ("energy") is conserved [3]. Depending on the nature of the coupling between the activity and the local density, the conservation law may [4] or may not [5] change the critical behavior from DP. Further, the non-DP behavior can belong to different universality classes such as conserved DP (C-DP), which describes stochastic sandpiles with linear coupling [6,7].

The models mentioned above have resisted an exact solution so far and have been studied extensively using numerical simulations and field-theoretic techniques. In this article, we introduce and study the steady state and the dynamics of a simple fixed energy sandpile model (FESM) for which some exact results can be found. As we shall see, the critical behavior of this model is different from that of the other known classes of FESMs.

In Sec. II, we define the model and study its steady state. In our model, only *one* particle hops out of a site to a nearest neighbor if the number of particles at that site exceeds a certain threshold. In the high-density active phase, this model defines a zero range process (ZRP) whose steady state is known to be of product measure form in all dimensions [8,9]. In the low-density absorbing phase, the steady state is not unique and depends on the initial conditions. We determine the critical behavior of this model by studying an order parameter and a correlation function close to the critical density. The time-dependent properties of this model are described in Sec. III. We argue that the relaxation dynamics are the same as that of a two-species annihilation process; in particular, we find a stretched exponential decay in the inactive phase, which has been seen numerically in other sandpiles also [10]. Our arguments are supported by Monte Carlo simulations.

II. THE MODEL AND ITS STEADY STATE

The model is defined on a d-dimensional hypercubic lattice of length L with periodic boundary conditions. Each site can hold an arbitrary number of particles with unit mass. We call a site active if the mass at this site is greater than m_{thr}, otherwise it is called inactive. A particle hops out of an active site to a nearest neighbor with equal probability. In continuous time, the hopping rate $u(m)$ at a site having m particles is given by

$$u(m)=\begin{cases}1/2d, & m>m_{thr},\\ 0, & m\leqslant m_{thr}.\end{cases} \tag{1}$$

The dynamics conserve the total density $\rho=M/V$ where M is the total number of particles in the system and $V\equiv L^d$ is the volume of the system. In one dimension with $m_{thr}=1$, this model can be mapped to a conserved lattice gas (CLG) model by regarding the sites as holes and mass as particle clusters. In the CLG language, a particle with one occupied neighbor can only hop to an empty nearest neighbor whereas the isolated particles do not move [11].

To see the phase transition, we define all the particles at a site to be immobile if the mass $m\leqslant m_{thr}$ at this site. For $m>m_{thr}$, the first m_{thr} particles are said to be immobile and the rest $m-m_{thr}$ mobile. If the initial number of particles is less than $m_{thr}V$, a mobile particle diffuses around till it reaches a site with $m<m_{thr}$ whereupon it becomes immobile. Thus, the number of mobile particles eventually becomes zero. On the other hand, if the total number of particles in the initial state exceeds $m_{thr}V$, then the number of mobile particles decreases until it reaches $M-m_{thr}V$. Thus, there is a phase transition at the critical density $\rho_c=m_{thr}$ from an inactive phase with only immobile particles to an active one with a finite number of mobile particles, as the total density is increased. For sake of simplicity, we choose $m_{thr}=1$ in the following discussion.

A. The active phase

We first note that the probability $P_k(0,t)$ of having $m=0$ at site k at time t is a monotonically decreasing function in time since

$$\frac{\partial P_k(0,t)}{\partial t}=-\frac{1}{2d}\sum_{\delta}\sum_{m\geqslant 2}P_{k,k+\delta}(0,m,t), \tag{2}$$

where $P_{k,k+\delta}(m',m,t)$ is the joint probability that the site k and its neighbor $k+\delta$ have mass m' and m, respectively, at time t. In the active phase, since $\rho>1$, it follows that the probability $P(0)$ of having an empty site is zero. It is then easy to check that the condition of detailed balance holds with the steady-state distribution $P(C \equiv \{m_1, \ldots, m_V\})$ being equally likely. Since the number of ways in which $M-V$ particles can be partitioned in V cells is given by $Z = \binom{M-1}{V-1}$ [12], we have $P(C) = 1/Z$. Using a similar reasoning, the mass distribution $P(m) = \binom{M-m-1}{V-2}/Z$ can be obtained. Thus, the activity $S = \Sigma_{m \geqslant 2} P(m) = (M-V)/(M-1)$. Near the critical point, $S \sim (\rho - \rho_c)^{\beta}$ with $\beta = 1$ in all dimensions. Intuitively, as $\rho \rightarrow \rho + 1/V$, due to the absence of empty sites, the newly added particle typically becomes a mobile particle, leading to a linear growth of activity. This argument fails for the models studied in [4] in which more than one particle can leave an active site, thus creating empty sites. In such cases, one may expect the activity to grow nonlinearly.

Since $P(0) = 0$ for our model, the exponent β is expected to be unity even if $u(m)$ is mass dependent for $m > m_{thr}$. This can be shown by realizing that the active phase is a special case of ZRP for which $P(C) \sim \Pi_{k=1}^{V} f(m_k)$ where the marginal $f(m_k)$ is given by

$$f(m_k) = \begin{cases} \displaystyle\prod_{n_k=2}^{m_k} 1/u(n_k), & m_k \geqslant 2, \\ 1, & m_k = 0, 1. \end{cases} \tag{3}$$

The distribution $P(m) = v^m f(m)/Z$ is obtained by using the measure $P(C)$ above with fixed density constraint. Here the normalization constant $Z = \Sigma_{m \geqslant 1} v^m f(m)$ and the fugacity v is determined by the conserved particle number condition which can be written as

$$\rho - 1 = \frac{1}{Z} \sum_{m \geqslant 2} (m-1) v^m f(m). \tag{4}$$

Close to the critical density, the fugacity $v \rightarrow 0$ and we can approximate $\rho - 1$ by the first term in the sum above. Using this in the expression for the activity $S = \Sigma_{m \geqslant 2} v^m f(m)/Z$ for $v \rightarrow 0$, it follows that $\beta = 1$.

Typically, the density-density correlation function $C(r, \rho) = \langle m_0 m_r \rangle - \rho^2$ decays exponentially with r and can be used to define a static correlation length $\xi \sim (\rho - \rho_c)^{-\nu_{\perp}}$. Since this phase has product measure, the correlation function $C(r, \rho)$ is zero in the thermodynamic limit for all r. This implies that the correlation length $\xi_{>} = 0$ and the exponent $\nu_{\perp}^{>}$ is undefined in this phase.

B. The inactive phase and the critical point

For $\rho < \rho_c$, the system is in the inactive phase and the number of absorbing states is given by $\binom{V}{M}$. Unlike the active phase, here the steady state is neither unique nor has product measure even if the initial state has. To see this, let us prepare an initial state in which a site is occupied by a monomer with probability p_1, a dimer with probability p_2, and remains empty with $p_0 = 1 - p_1 - p_2$. Then, for $p_1 = 0$, the final state will have one-clusters with only an even number of ones, whereas, for $p_1 \neq 0$, the configurations with an odd number of ones are also allowed. Thus different initial conditions give different solutions for $P(C)$ so that the steady state is not unique. Further, this is clearly not a product measure state for which the weight of all configurations at a fixed density is same.

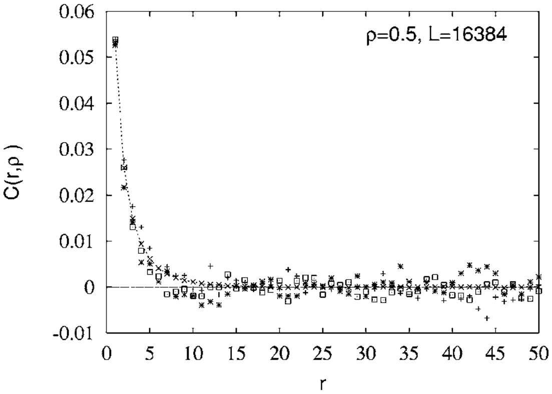

FIG. 1. Plot of $C(r, \rho)$ vs r for $\rho < \rho_c$ in one dimension with Poisson distributed initial mass. The data shown with points corresponds to different initial seeds for the random number generator. The data averaged over 1000 initial conditions is shown with broken line.

Using Monte Carlo simulations in one dimension, we measured the correlation function $C(r, \rho) = \langle m_0 m_r \rangle - \rho^2$ where m is either 0 or 1. Since the steady state depends on the initial condition, the angular brackets denote spatial averaging for a given initial condition. We initially distribute particles independently at each site with the mass chosen from a Poisson distribution. As shown in Fig. 1, the behavior of the correlation function $C(r, \rho)$ when averaged over such initial conditions is similar to when only spatial averaging is done for a fixed initial configuration. Therefore, in the following discussion, we will carry out ensemble averaging as well.

The data in the inset of Fig. 2 for $C(r, \rho)$ at various densities for a large, one-dimensional system supports the claim that the product measure does not hold in this phase. For

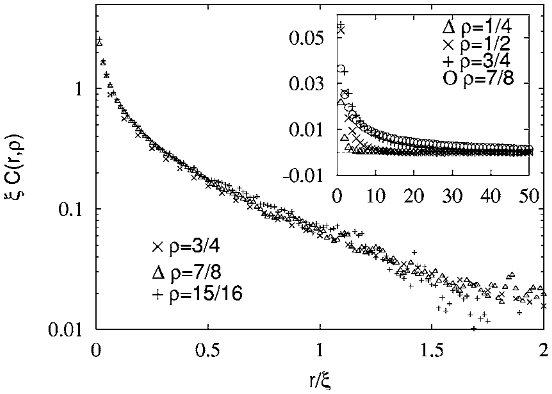

FIG. 2. Inset: Plot of $C(r, \rho)$ vs r in one dimension for $\rho < \rho_c$ with $L = 16\,384$ to show that the product measure does not hold in the inactive phase. Main: Data collapse for the scaled correlation function $\xi C(r, \rho)$ vs. r/ξ with $\xi = (1-\rho)^{-2}$.

densities away from unity, the correlation function decays fast and as one approaches the critical point, the decay gets slower. The magnitude of $C(r,\rho)$ at small r drops for $\rho \to 1$ since, at the critical density, we must obtain $C(r,\rho_c)=0$. Also, since the total mass is conserved, we have

$$\langle M m_0 \rangle = L\rho^2 = \langle m_0^2 \rangle + \langle m_0 m_1 \rangle + \cdots + \langle m_0 m_{L-1} \rangle . \qquad (5)$$

Thus, the correlation function obeys the sum rule $\Sigma_r C(r,\rho) = 0$ due to which $C(r,\rho)$ is not positive for all r. Finally, as shown in Fig. 2, a data collapse for$C(r,\rho)$ for densities close to the critical density can be obtained if we assume

$$C(r,\rho) \approx \xi_<^{-1} g(r/\xi_<), \quad \xi_< = (1-\rho)^{-2}. \qquad (6)$$

The above scaling form is consistent with vanishing $C(r,\rho)$ at the critical point. Thus, the static correlation length $\xi_< \sim (\rho_c - \rho)^{-\nu_\perp^<}$ diverges at the critical point with exponent $\nu_\perp^< \approx 2$.

III. TIME-DEPENDENT BEHAVIOR

Starting from an initial state in which particles are Poisson distributed, we study the temporal decay of the activity at and below the critical point. The activity $S(t)$ at time t obeys the following equation:

$$\frac{\partial S(t)}{\partial t} = \frac{1}{2d} \sum_\delta \left(\sum_{m \geq 2} P_{k,k+\delta}(1,m,t) - \sum_{m \geq 1} P_{k,k+\delta}(2,m,t) - P_{k,k+\delta}(2,0,t) \right). \qquad (7)$$

The first term on the right-hand side (RHS) represents the gain in the activity when a particle hops out of an active site to an inactive site, whereas the last two loss terms correspond to a particle leaving an active site with two particles. At low densities, since the probability of having large mass is exponentially small, the first two terms on the RHS of Eq. (7) can be ignored and $S(t)$ can be approximated by $P(2,t)$. Denoting $m=0$ by A, $m=1$ by ϕ, and $m=2$ by B, the last term describes a two-species annihilation reaction $A+B \to \phi$ where A is a static and B is a diffusing species. A similar mapping to reaction-diffusion system for the models in [4] involves more complicated reactions due to the creation of empty sites. In the following, we will denote the number of species A and B at time t by $n_A(t)$ and $n_B(t)$, respectively.

At the critical density $\rho_c = 1$, we expect that $n_B(0) \approx n_A(0)$ and in a finite domain of length ℓ, they differ by $\pm O(\ell^{d/2})$. Since the species B performs diffusive random walk, at large times in an infinite system, we have [13]

$$S(\rho_c^\infty, t) \sim \begin{cases} t^{-d/4}, & d < 4, \\ t^{-1}, & d \geq 4, \end{cases} \qquad (8)$$

where ρ_c^∞ refers to the critical density in the thermodynamic limit. For a finite system of size L, the activity S at density ρ_c^∞ is expected to be of the scaling form,

$$S(\rho_c^\infty, t, L) \approx t^{-\theta} H(t/L^z), \qquad (9)$$

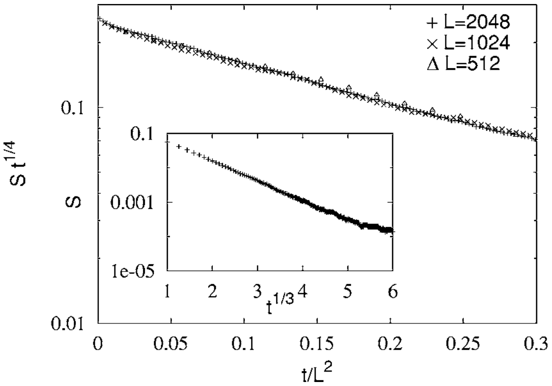

FIG. 3. Data collapse for the scaled activity $t^{1/4} S(t,L)$ vs t/L^2 at $\rho = \rho_c$ in $1d$ in accordance with Eq. (9). The inset shows the temporal decay of $S(t)$ for $\rho < \rho_c$ as in Eq. (10).

where the scaling function $H(x)$ is a constant for $x \ll 1$ and grows as x^θ for $x \gg 1$ [4]. In other words, the activity typically decays as a power law in time to a system-size-dependent constant $L^{-z\theta}$. As shown in Fig. 3, in one dimension, the activity S is of the scaling form in Eq. (9) with $z \approx 2$ and the exponent θ given by Eq. (8). However, the scaling function $H(x)$ *decays* exponentially for $x \gg 1$ as seen by the linear decay of scaled activity on the semilog scale in Fig. 3. In two dimensions, our simulations support the scaling behavior Eq. (9) with the scaling function decaying as a stretched exponential. The finite size scaling of the activity $S \propto L^{-z\theta}$ in the steady state does not hold since the exact expression for $S=(M-V)/(M-1)$ gives $\rho_c = 1$ for *any* L (unlike other sandpiles where ρ_c is expected to be size dependent).

For $\rho \ll \rho_c$, we expect $n_B(0) \ll n_A(0)$ so that n_A does not change appreciably due to the annihilation reaction and we are led to consider the problem of diffusing B species in the presence of static traps A. In this case, at large times, the density $S(t)$ decays as a stretched exponential [14],

$$S(t) \sim \exp[-(t/t_d)^\alpha], \quad \text{for } \rho < \rho_c, \qquad (10)$$

where $\alpha = d/(d+2)$ and $t_d \sim (1-\rho)^{-2/d}$. This slow decay arises due to large trap-free regions which are rare but enhance survival significantly. The inset of Fig. 3 shows the decay of $S(t)$ in one dimension in accordance with Eq. (10). This behavior of slow relaxation to the steady state is similar to that observed numerically in various stochastic FESMs in [10].

We next consider the effect of a perturbation in the steady state by considering the correlation function $G(\vec{r},t) = \langle \eta(0,0)\, \eta(\vec{r},t) \rangle$ where $\eta(\vec{r},t)$ is one if the site $\vec{r}$ is active at time t and zero otherwise. We perturb the steady state by constructing an initial condition with a single mobile particle placed at the origin. At the critical point, this particle executes a random walk and the correlation function $G(\vec{r},t)$ is equal to the probability that the walker is at $(\vec{r},t)$ so that

$$G(\rho_c, r, t) \approx \frac{1}{(2\pi t)^{d/2}} \exp(-r^2/2t). \qquad (11)$$

In the absorbing phase, there is a finite density $1-\rho$ of the vacant sites and the typical distance r_0 between them scales as $(1-\rho)^{-1/d}$. Therefore, the single mobile particle at the origin diffuses for $t \ll r_0^2$ so that the correlation function $G(\vec{r},t) \sim e^{-r^2/2t}, r < r_0$, increases with time. However, for $t \gg r_0^2$, the mobile particle gets trapped and $G(\vec{r},t)$ decays as in Eq. (10). Usually, the autocorrelation function $G(0,t)$ is expected to decay exponentially, i.e., $G(0,t) \sim e^{-t/\tau}$ where $\tau \sim (\rho_c - \rho)^{-\nu_\parallel}$ and the exponent $\nu_\parallel$ obeys the scaling relation $\nu_\parallel = \nu_\perp z$ [1]. The stretched exponential behavior of $G(0,t)$ in our model implies that $\nu_\parallel^<$ is infinite. Alternatively, one can define this exponent via $t_d \sim (\rho_c - \rho)^{-\nu_\parallel^<}$ with $\nu_\parallel^< = 2/d$. Since, in our model, $z \approx 2$ and the correlation length exponent $\nu_\perp^< \approx 2$ in one dimension, either definition of $\nu_\parallel^<$ is inconsistent with the above scaling relation and we conclude that it does not hold in the absorbing phase.

IV. CONCLUSION

We introduced a sandpile model whose simplicity allowed us to determine various static and dynamic exponents exactly. These exponents differ from those of the C-DP class for which, in one dimension, $\beta \approx 0.29, \nu_\perp^> \approx 1.33, z \approx 1.55$, and $\theta \approx 0.14$ [7]. We considered the correlation lengths $\xi^>$ and $\xi^<$ defined above and below the critical density, respectively, and found that while $\xi^>$ is zero, $\xi^<$ diverges as $\rho \to \rho_c -$. However, a further detailed study of the nontrivial correlations present in the inactive phase would be interesting. We also studied the temporal properties of this system by relating it to a well-studied reaction-diffusion system via an argument similar to that used in [15] to study avalanche size distribution in a sandpile model.

One can also consider the biased version of the above model in which a particle moves preferentially to a nearest neighbor. The steady state of this model is the same as that for the symmetric case discussed above. For the asymmetric case, using the known results for the two-species annihilation problem with the B species drifting with a nonzero speed, we expect that at the critical point, $S(t)$ decays as a power law with an exponent $d/2$ for $d \leqslant 2$ and exponentially in the subcritical regime [16]. Our numerical results are consistent with the argument.

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